Overcoming Borrowing Stigma: The Design of Lending-of-Last-Resort Policies

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Abstract

How should the government effectively provide liquidity to banks during periods of financial distress? During the most recent financial crisis, banks avoided borrowing from the Fed's Discount Window (DW) but bid more in its Term Auction Facility (TAF), although both programs share similar requirements on participation. Moreover, some banks paid higher interest rates in the auction than the concurrent discount rate. Using a model with endogenous borrowing stigma, we explain how the combination of the DW and the TAF increased banks' borrowings and willingnesses to pay for loans from the Fed. Using micro-level data on DW borrowing and TAF bidding from 2007 to 2010, we confirm our theoretical predictions about the pre-borrowing and post-borrowing conditions of banks in different facilities. Finally, we discuss the design of lending-of-last-resort policies.

Keywords: discount window stigma, auction, adverse selection, lending of last resort

JEL: G01, D44, E58

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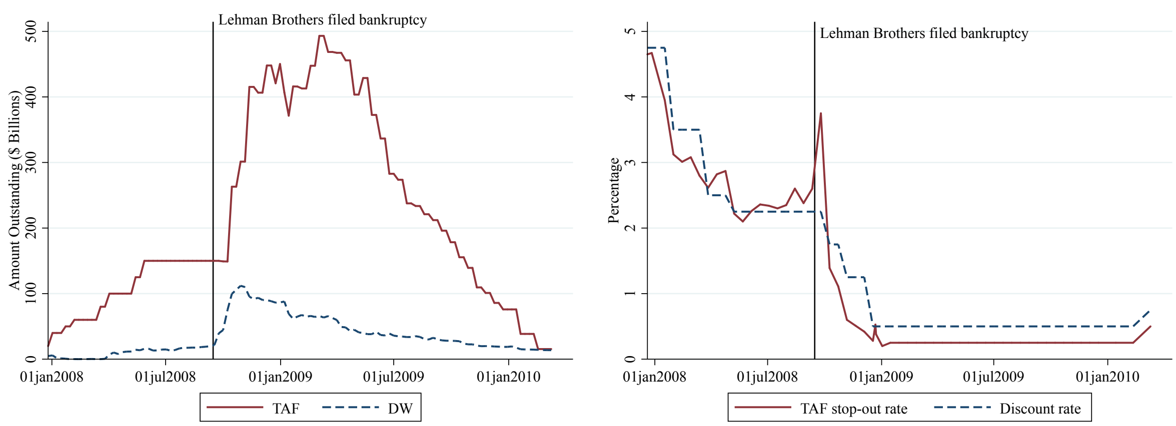
1 Introduction

Financial crises are typically accompanied with liquidity shortage in the entire banking sector. How should the central bank lend to depository institutions during such episodes? The answer is not obvious. The discount window (DW) has been the primary lending facility used by the Federal Reserve, but it was severely under-used when the interbank market froze in the onset of the financial crisis in late 2007. A main reason for such under use is believed to be the stigma associated with DW borrowing: tapping the discount window conveys a negative signal about the borrowers' financial conditions to their counterparties, competitors, regulators, and the public.[1] As suggestive evidence, banks have regularly paid more for loans from the interbank market than they could readily get from the DW (Peristiani, 1998; Furfine, 2001, 2003, 2005).

Figure 1: Borrowing amount and borrowing rates in DW and TAF from 2008 to 2010

(a) Borrowing Amount (b) Borrowing Rates



In response to the credit crunch and banks' reluctance to borrow from the DW, the Fed created a temporary program, the Term Auction Facility (TAF), in December 2007. The TAF held an auction every other week, providing a pre-announced amount of loans with *identical*

[1] Although the Fed does not disclose publicly which institutions have received loans from the DW, the Board of Governors publishes weekly the total amount of DW lending by each of the twelve Federal Reserve Districts. Therefore, a surge in total DW borrowing could send the market scrambling to identify the loan recipients. Because of the interconnectedness of the interbank lending market, it is not impossible for other banks to infer which institutions went to the Discount Window. Market participants and social media can also infer from other activities.

loan maturity, collateral margins, and eligibility criteria as the DW. Surprisingly, the TAF provided much more liquidity than the DW: Figure 1a shows that the outstanding balance of TAF borrowing could be five times as much as DW borrowing during 2007-2010. Moreover, interestingly, banks sometimes paid a higher interest rate to obtain liquidity through the auction: Figure 1b shows that the stop-out rate – the rate that cleared the auction – was higher than the concurrent discount rate – the rate readily available in the DW – in 21 out of the 60 auctions, especially from March to September 2008.

This episode suggests the importance of the design of emergency lending programs to effectively cope liquidity shortage. More specifically, it raises a series of questions about the lending-of-last-resort policies. Why could the TAF overcome the stigma and generate more borrowing than the DW? Shouldn't the similar stigma also prevent banks from participating in the TAF? How did banks decide to borrow from the DW and/or the TAF? Was there any systematic difference between the banks that borrowed from two facilities? How to further improve the program? The answers to these questions remain unclear, even to policy makers involved (Armantier and Sporn, 2013; Bernanke, 2015).

This paper provides a comprehensive analysis of lending of last resort in the presence of borrowing stigma. Specifically, we introduce a dynamic model in which banks have private information about their financial conditions. Weaker banks have more urgent liquidity need and enjoy higher borrowing benefits. Two lending facilities are available. An auction is held once to allocate a set amount of liquidity, and the DW is available before and after the auction. Borrowing from each facility suffers from a stigma cost, which is endogenously determined by the financial conditions of participating banks.

In equilibrium, banks self select into different programs. Since the DW always guarantees lending, the weakest banks borrow from it immediately, because they are desperate for liquidity and cannot afford to wait. Stronger banks, in contrast, are lured to participate in the auction because the potential of borrowing cheap renders the auction more attractive than the DW. Their liquidity needs are not that imperative and they value lower expected price in the auction more than their weaker counterparts. Among the banks who lose in TAF, relatively weaker ones might still borrow from the DW. Finally, the strongest banks do not borrow at all. Among the banks who participate in the TAF, some may bid higher than the discount rate because they would like to avoid the discount window stigma brought by the weakest banks. As a result, the clearing price in the auction may exceed the discount rate.

In our model, the introduction of TAF in addition to DW could increase liquidity provision through three channels. First, by setting a low reserve price in the auction, the TAF attracted relatively strong banks to participate and take their chances of borrowing cheap. Second, participating banks can internalize any stigma cost associated with the TAF by adjusting their bids, which endogenously leads to a positive payoff if they win. Third and finally, due to selection, the auction stigma is endogenously lower than the discount window stigma. We show the introduction of TAF expands the set of the banks who try to and may obtain liquidity, thus potentially increasing the supply of short-term credit to the economy.

We use micro-level data on DW borrowing and TAF bidding to verify the model's prediction. We obtain two sets of empirical results. The first set of results confirms our prediction that banks opt into different borrowing programs. We find that (i) weaker banks – measured by tier-1 capital ratios – were more likely to tap the DW relative to the TAF and (ii) among the banks who participated in the TAF, those who submitted higher bids (and thus were more likely to be winners) pledged collaterals of lower quality and were more likely to bid again in subsequent auctions (a sign of weakness). The second set of results confirms our prediction that the stigma is different for different programs. Since the TAF schedule was announced weeks before the auction date, tapping the discount window right before the auction signals a bank's weakness. Using an event-study approach, we confirm that who borrowed from the DW within three days before an auction were associated with more negative subsequent abnormal returns in their stock prices.

Our paper improves the understanding of interventions during the financial crisis, and more specifically, contributes to the literature that studies government intervention in markets plagued by adverse selection (Philippon and Skreta, 2012; Tirole, 2012; Ennis and Weinberg, 2013; La'O, 2014; Lowery, 2014; Fuchs and Skrzypacz, 2015; Gauthier et al., 2015; Li et al., 2016; Ennis, 2017; Che et al., 2018). In these studies, either there is no explicit stigma cost in government-sponsored facility, or stigma is implicitly assumed as identical across all programs. Our paper models the stigma cost associated with DW and TAF differently and exploits banks' endogenous decisions on which facility to use.

This paper, to the best of our knowledge, is the first to combine micro-level data on DW borrowing and TAF bidding, and link them to information on banks' fundamentals. Existing papers on the discount window and the term auction facility are largely empirical and/or policy-oriented. Peristiani (1998); Furfine (2001, 2003, 2005) offer evidence that banks prefer the Federal Funds Market to the DW, suggesting the existence of the DW stigma. More recently, Armantier et al. (2015) show that more than half of the TAF participants submitted bids above the discount rate during the 2007-2008 financial crisis. McAndrews et al. (2017) and Wu (2011) study the effect of the TAF and conclude that it was effective in lowering Libor and reducing liquidity concern in the interbank lending market. Moore (2017) finds that the TAF had a benefit on the real economy. Cassola et al. (2013) study the financial crisis from the bidding data in the European central bank from January to December, 2017.

2 The Model

We introduce a three-date, two-period model with n banks in the economy. A period corresponds to a week in the real world. Figure 2 sketches the timing and sequence of events. Banks are endowed with illiquid assets that pays off R at $t = 2$. Shortly before the asset pays off, banks may be hit with liquidity shocks (θ). Between $t = 0$ and $t = 2$, they can borrow from one of the two facilities: discount window (DW) and term auction facility (TAF). Borrowing banks may be detected, in which case an endogenous penalty k is imposed.

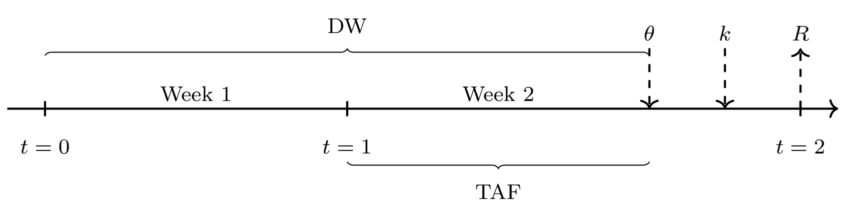


Figure 2: Timeline of the model

2.1 Preferences, Technology, and Shocks

All parties are risk neutral and do not discount future cash flows.

At $t = 0$, each bank is endowed with one unit of long-term, illiquid assets that will mature at $t = 2$. The asset generates cash flows R upon maturity but nothing if liquidated early. Shortly before $t = 2$, each bank may be hit with a liquidity shock a-la **?**. The size of the shock is normalized as one unit. Let $1 - \theta_i \in [0, 1]$ be the probability that the liquidity shock hits bank i, where θ_i follows $i.i.d.$ and has pdf $f(\cdot)$ on $[0, 1]$. Throughout the paper, we assume θ_i is private information and only known by the bank itself. We drop subscript i whenever no confusion arises. Type θ is also referred to as the financial strength. In reality, one can proxy θ as either banks' reserve of liquid assets or the level of its demand-able liabilities that can evaporate in a flash.

Before the liquidity shock hits, each bank has the opportunity to borrow. We will describe the choices of borrowing. For now, let r be the gross interest rate of a received loan. A loan will help the bank defray the liquidity shock and therefore brings net benefits $(1 - \theta) R$ at the cost of interest rate r. Finally, to capture the idea that earlier liquidity is more valuable, we assume the net benefits are discounted by a common factor δ if borrowing is accomplished in week 2. δ can be interpreted as the cost incurred when banks sell illiquid assets at fire-sale prices in order to satisfy immediate liquidity needs. Moreover, one can easily provide a microfoundation by introducing another liquidity shock that may hit at $t = 1$. To summarize, a bank's overall payoff to borrowing is $\pi = \pi_1(\theta, r) = (1 - \theta) R - r$ if it borrows in week 1 and $\pi = \pi_2(\theta, r) = \delta(1 - \theta) R - r$ if it borrows in week 2. As it becomes clear later on, the specific functional form of the borrowing benefit won't matter. What matters is such benefits are lower for stronger banks and/or if the interest rate is higher.

We describe the two lending facilities in the next subsection.

2.2 Borrowing Facilities

We will describe an extension in which the interbank market is well-functioned. In the basic model, any bank is only allowed to borrow from either the discount window or the term auction facility.

Discount Window

We model the discount window as a facility that offers loans at a fixed interest rate r_D. r_D is also referred to as the discount rate, which is exogenously set by the Federal Reserve. Since a bank can always borrow from the discount window with certainty, the net borrowing benefit is $\pi(\theta, r_D)$.

Term Auction Facility

The term auction facility is characterized by (m, r_A), where $m < n$ is the total units of liquidity offered and r_A is the reserve price. In an auction, banks who decide to participate simultaneously submit their sealed bids.[2] Bid β_i specifies the interest rate bank i is willing to pay. The bid needs to be higher than r_A.

After receiving all the bids, the auctioneer ranks them from the highest to the lowest. The auction takes a uniform-price format: all winners pay for the same interest rate while losers do not pay anything. If there are fewer bids than the units of liquidity provided, each bidder receives a loan and pays r_A. If there are more bidders than the total offering liquidity, each of the m highest bidders receives one unit of liquidity by paying the highest *losing* bid. In this case, the highest losing bid is also called the stop-out rate s, which is the clearing price at which aggregate demand in the auction matches the aggregate supply. Formally, let β_1 be the highest bid and β_l be the lowest one in the case with l bidders in total. If $l \leq m$, bidding banks each receive a loan by paying $s = r_A$. If $l > m$, the m highest bidding banks each receive one unit of liquidity by paying $s = \beta_{m+1}$. The remaining $l - m$ banks do not pay anything and, of course, do not receive any liquidity either.

Let $w(\theta, \beta(\theta))$ be the equilibrium probability that bank θ can win the auction by bidding $\beta(\theta)$. We will focus on symmetric strategies in bidding and therefore can write $w(\theta, \beta(\theta))$ as $w(\theta)$ without loss of generality. Also let $b(\theta)$ be the expected payment that bank θ pays conditional on winning the auction. The expected net borrowing benefit is $w(\theta)\pi(\theta, b(\theta))$.

We have essentially modeled the TAF auction as an extended second-price auction: all winning parties pay the highest losing bid. In reality, TAF is closer to an extended first-

[2] We assume that each bank is restricted to bid for only one unit of liquidity so that no secondary market exists for the auction-allocated liquidity.

price auction: all winning banks pay the lowest winning bid. We can show that both setups generate equivalent payoffs and borrowing decisions, by the Revenue Equivalent Theorem (Myerson, 1981). We present the analysis with the extended second-price auction, because it is a weakly dominant strategy for each bank to simply bid the maximum interest rate it is willing to pay.[3]

2.3 Detection and Stigma

A key reason that banks were reluctant to borrow from the lender of the last resort is stigma. Indeed, detected borrowing may signal financial weakness to counter-parties, investors, and regulators. Although θ is private information, the public can still make inference based on whether the bank has borrowed and if so, which facility the bank has used. We assume that upon detection, the public can perfectly tell whether the borrowing has been achieved through the discount window or the auction. In the basic model, we assume the public cannot tell *when* the bank has borrowed from the discount window. Later on, we will show that none of our results is driven by the specific assumptions on detection.

Let Θ_D, Θ_A, and Θ_N be the set of banks who have been detected borrowing from DW, TAF, and not borrowing at all. Also, let $\Theta_\varnothing$ be the set of banks who are not detected of any activity. Let G_D, G_A, G_N, and $G_\varnothing$ be the (ex-post) cumulative distributions of banks on these sets that are consistent with the equilibrium borrowing decisions. We capture this notion of stigma in a parsimonious way. Specifically, we assume that after all the borrowings are accomplished, banks that have successfully borrowed may be detected independently with probability p, after which a penalty will be imposed. This penalty can be understood as cost in bank's reputation, cost in finding counterparties, or runs and increasing withdrawals by creditors. Let it be $k(\theta, G_\omega)$, where $\omega \in \{D, A, N\}$ is an indicator function for whether the bank has borrowed through the discount window, the auction, or not borrowing at all. Clearly, the stigma cost is higher when the perceived set of borrowing banks are worse: $k(\theta, G) > k(\theta, G')$ if G is strictly first-order stochastically dominated by G'. In the baseline model, we eliminate the dependence of stigma cost on a bank's private type and instead

[3] In contrast, in the first-price auction, banks shade their bids that depend on the liquidity supply and other participating banks

assume it only depends on its borrowing facility $\omega \in \{D, A, N\}$. In other words, $k(\theta, G_\omega) = k(G_\omega) \equiv k_\omega$.

2.4 Equilibrium

A type θ bank's strategy can be succinctly described by $\sigma(\theta) = (\sigma_D(\theta), (\sigma_A(\theta), \beta(\theta)))$, where $\sigma_\omega(\theta),\ \omega \in \{D, A\}$ is the probability of borrowing from each facility and $\beta(\theta)$ is its bid if it participates the auction. Given strategies $\sigma(\cdot)$, beliefs about the financial situation can be inferred by the Bayes' Rule. In this case, we say aggregate strategies $\sigma(\cdot)$ generate posterior beliefs G. Note that we have restricted each bank's strategy to be symmetric so that $\sigma(\cdot)$ only depends on θ.

Definition 1. $(\sigma^*(\cdot), G^*)$ *form a sequential equilibrium Perfect Bayesian equilibrium) if*

1. *each type θ bank's strategy $\sigma^*(\theta)$ maximizes its expected payoff given belief system G^*;*

For the remainder of this paper, we will refer to the sequential equilibrium as the equilibrium.

2.5 Parametric Restrictions and Equilibrium Refinement

We introduce parametric restrictions which not only resemble the reality but also eliminate the unrealistic equilibria.

Clearly, the best bank (bank of type 1) has no intention to borrow at all. It only pays a price and stigma cost but has no benefit from borrowing. The next assumption requires a wide span of banks. In particular, the borrowing benefit of the worst bank (bank of type 0) is sufficiently high.

Assumption 1. $\delta R - r_D - k(G_\omega \equiv \{0\}) > 0$.

Finally, we make assumptions on refinement of off-equilibrium path. If one of the two facility is not used on equilibrium path, market participants will apply intuitive criterion to form beliefs on a bank's type (Cho and Kreps, 1987).

3 Equilibrium Solutions

3.1 Benchmark

We present two benchmark cases before moving on to the solution of the entire model.

Equilibrium with only DW

We start by examining the equilibrium when the government only sets up the discount window (finite r_D and $m = 0$). Clearly, no bank would ever want to borrow from the discount window in the second period. We show the equilibrium is characterized by one threshold: weaker banks borrow from discount window and stronger banks do not borrow at all.

Proposition 1. *If r_D is finite and $m = 0$, there exists an equilibrium characterized by θ^{DW}*

1. *Banks between $\left[0, \theta^{DW}\right]$ borrow from the discount window.*
2. *Banks between $\left[\theta^{DW}, 1\right]$ do not borrow at all.*

Note a bank whose type is $\theta = 1$ never borrows. It knows a liquidity shock could never occur and therefore never need the liquidity, but borrowing incurs a gross interest rate r_D as well as the stigma k_D. We also assume a bank whose type satisfies $\theta = 0$ always borrows.

Equilibrium with only TAF

Next, we examine the equilibrium when the government only sets up the auction (infinite r_D and $m > 0$). We show the equilibrium is characterized by one threshold: weaker banks bid in the auction and stronger banks do not borrow at all.

Proposition 2. *If r_D is infinite and $m > 0$, there exists an equilibrium characterized by θ^{TAF}*

1. *Banks between $\left[0, \theta^{TAF}\right]$ bid in TAF.*
2. *Banks between $\left[\theta^{TAF}, 1\right]$ do not bid at all.*

We end this subsection with a comparison between the two benchmark cases and highlight the important forces behind such a comparison. Clearly, lower discount rate and higher reserve price in the auction will enable DW borrowing more attractive. On the other hand, if the borrowing benefit gets high and discounting gets low, banks are less willing to wait for TAF.

Corollary 1. *$\theta^{DW} - \theta^{TAF}$ increases with r_A and R, decreases with r_D, and δ.*

3.2 Equilibrium with DW and TAF

In this subsection, we solve for the equilibrium when both discount window and term auction facility are available to use. We will first describe a bank's bidding strategy in TAF, followed by its incentives in choosing between DW and TAF. Our result shows that relatively stronger banks have more incentives to bid in TAF rather than borrow immediately from DW, which is the key force behind the equilibrium segregation.

Let us start by describing a bank's bid in the auction. In general, a bank's bidding strategy depends on its plan after losing in the auction: it can either borrow from the DW in the second period or not to borrow at all. Clearly in this case, the incentive to borrow declines with bank's financial strength.

Lemma 1. *In any equilibrium, among banks who still seek funding in period 2, $\theta < \theta_2$ will borrow from the discount window.*

Let $\beta^D(\theta)$ be bank-θ's bid if it plans to borrow from discount window after losing the auction. Let $\beta^N(\theta)$ be its bid if it doesn't plan to borrow after losing the auction. Given that a bank's bid does not (directly) affect its payment conditional on winning the auction, a bank bid its own willingness to pay (WLP), as follows.

Lemma 2.

$$\beta^D(\theta) = r_D + (k_D - k_A) \tag{1}$$

$$\beta^N(\theta) = \delta(1-\theta)R - k_A \tag{2}$$

Note that $\beta^D(\theta)$ does not depend on θ. In other words, any bank who plans to go to the discount window bids up to the same amount, which equals the sum of r_D, the discount rate, and $(k_D - k_A)$, the stigma of discount window relative to TAF. Intuitively, these banks will always borrow in equilibrium, from either the discount window or TAF. Therefore, since the discount window charges the same rate to all borrowers and the stigma cost is also homogeneous across all borrowers from the same facility, their WLPs are also the same. In the most general case where k_ω depends both on the borrowing decision $\omega \in \{D, A\}$ and a bank's own financial strength θ, $\beta^D(\theta)$ will decrease in θ as long as $k(\theta, G_D) - k(\theta, G_A) > 0$ for any θ. On the other hand, $\beta^N(\theta)$, however, does depend on θ. Among these banks, weaker ones have higher WLPs because they have stronger demand for liquidity but will not borrow if they lose in TAF.

Proposition 3 is a main result of this paper. It describes the incentive to borrow from DW1 against participating the auction. In particular, it shows the skimming property that stronger banks are more willing to wait for TAF relative to weak ones.

Proposition 3 (Skimming property)**.** *Let $u_1(\theta)$ be bank θ's expected equilibrium payoff if it borrows from the discount window in period 1, and $u_A(\theta)$ be its expected payoff if it bids in auction. In any equilibrium, $u_1(\theta) - u_A(\theta)$ decreases with θ.*

Intuitively, auction introduces uncertainty in terms of whether a bidding bank is able to borrow and if so at what price. Specifically, it introduces one mechanism that enables a bank to borrow at a low rate–lower than its own willingness to pay, at the cost of potentially failing to borrow (for banks $\theta \in [\theta_2, 1]$) or delaying to borrow (for banks $\theta \in [0, \theta_2]$). This cost of not borrowing (or delayed borrowing) is lower for stronger banks because their borrowing benefits are lower. Therefore, they are more inclined to participate in the auction and take advantage of the opportunity to borrow when rates are sufficiently low. In this case, auction is able to separate borrowers into two groups –the so-called "single-crossing" condition. Mathematically, a bank in $[0, \theta_2]$ will always borrow even if it chooses TAF: it will turn to the discount window in week 2 in the event of losing in TAF, in which case the cost of delay is $(1-\delta)(1-\theta)R$, decreasing in θ. A bank in $[\theta_2, 1]$ no longer borrows if it loses in auction, with the cost of failing to borrowing being $(1-\theta)R$.

It is worthwhile to point out that our results on separation does not depend on the assumption that delaying cost is bigger for weaker banks. In the appendix, we present another version of the model in which cost of delay is homogeneous across all banks and show all results carry through. Moreover, we would like to emphasize that not any mechanism that offers a tradeoff between probability of winning and price paid can always separate borrower. To see this, note that a bank's overall payoff has a total of three components that varies with θ. First, a stronger bank has lower borrowing benefits: $\frac{d(1-\theta)R}{d\theta} = -R < 0$. Second, in equilibrium, a stronger bank is less likely to win in the auction. However, conditional on winning in the auction, however, it pays less in expectation. When a bank bids optimally, it is indifferent between raising the bid to increase the winning probability and paying more conditional on winning. Therefore, the last two effects exactly cancel out on each other As a result, the overall effect is simply the decreasing benefits of borrowing times the probability of winning in the auction: $-R\left[1-H\left(\theta\right)\right]$. Next, let us consider a mechanism $\left(w\left(\theta\right), b\left(\theta\right)\right)$ where $w\left(\theta\right)$ is the probability of receiving one unit of liquidity and $b\left(\theta\right)$ is the price paid. Let a bank's payoff in participating this mechanism be $u_M\left(\theta\right)$.

$$u_1\left(\theta\right) - u_M\left(\theta\right) = w\left(\theta\right)\left[b\left(\theta\right) + k_M + \delta - r_D - k_D\right] + \left[1 - w\left(\theta\right)\right]\left[\left(1-\theta\right)R - r_D - k_D + k_N\right].$$

By taking derivatives w.r.t. θ, clearly the overall effect is ambiguous.

Given Proposition 3, the equilibrium with be that weaker banks choose to borrow from discount window in week 1, whereas stronger banks borrow bid in auction. Among banks who lose in the auction, relatively stronger ones will still go to the auction.

Theorem 1. *An equilibrium exists and is characterized by* $\{\theta_D, \theta_2, \theta_A\}$

1. *Banks* $\theta \in [0, \theta_D]$ *borrow directly from week 1's DW.*

2. *Banks with a financial condition between* θ_D *and* θ_A *behave as follows.*

   (a) *If* $\Delta(\theta_2|H(\cdot|\theta_2)) \leq 0$*, then* $\theta_D < \theta_2$*. Banks with a financial condition* θ *between* θ_D *and* θ_2 *bid* $\beta\left(\theta\right) = \rho + p(k_D - k_A)$ *in the auction, and borrow from week 2's DW if they lose in the auction. Banks with a financial condition between* θ_2 *and*

θ_A bid $\beta(\theta) = \delta b(\theta) - p(k_A - k_N)$ in the auction, and choose not to borrow from week 2's DW if they lose in the auction.

(b) If $\Delta(\theta_2|H(\cdot|\theta_2)) > 0$, then $\theta_D \geq \theta_2$. Banks with a financial condition θ between θ_D and θ_A bid $\beta(\theta) = \delta b(\theta) - p(k_A - k_N)$ in the auction, and choose not to borrow if they lose in the auction.

3. Banks $\theta \in [\theta_A, 1]$ do not borrow at all.

Proposition 3 immediately imply that the stigma associated with discount window borrowing exceeds that with TAF.

Corollary 2. *In any equilibrium, $k_D > k_A$ so that discount window carries a higher stigma than term auction facility.*

Liquidity Provision

The following proposition studies how the introduction of TAF changes the liquidity provided in equilibrium.

Proposition 4. *In equilibrium, $\theta_A > \theta^{DW} > \theta_D$.*

The result $\theta_A > \theta^{DW}$ clearly implies that the introduction of TAF expands the set of banks that *may* receive liquidity. However, $\theta^{DW} > \theta_D$ so that the set of banks will guarantee to receive liquidity actually gets smaller. Intuitively, the chances of borrowing at low rate at TAF induce some banks that would borrow from DW to wait for the auction. In Appendix A.1, we derive the detailed expressions for total expected liquidity provided by the government, which includes the amount of liquidity provided by the discount window and TAF. As illustrated by the following numerical example, the overall effect is ambiguous.

The government can control the size of δ by choosing the frequency of holding TAF.

Numerical Illustration We pick the following parameter values: $p = 0.8$, $n = 20$, $r_D = 1.3$, $R = 5$, $m = 1$, $r_A = 1$, and $\delta = 0.45$. Moreover, we assume uniform distribution so that $f(\cdot) \equiv 1$ and $k(\theta) = p(1-\theta)$. In this case, the equilibrium thresholds are $\theta_D = 0.62$ and $\theta_A = 0.68$. In other words, banks whose types are in $[0, 0.62]$ borrow from discount window directly, whereas banks whose types in $[0.62, 0.68]$ bid in the auction for the unit of liquidity. Indeed, $\theta_A > \theta^{DW}$ so that TAF expands the set of banks that may receive liquidity.

If TAF were not available, the threshold $\theta^{DW} = 0.66$, which gives rise to total expected liquidity provision of 13.2. With TAF, the total expected liquidity actually gets reduced to 12.41. The reason is $\theta^D < \theta^{DW}$ so that the set of banks that will borrow from discount window in week 1 drop significantly. After TAF, discount window only lends to 12.40 unit of liquidity, compared to 13.20 without TAF. However, the set of banks who bid in TAF is still limited and therefore the expected liquidity provided in TAF does not make up for the shortfalls in discount window liquidity.

4 Empirical Analysis

In this section, we offer some empirical evidence that is consistent with the prediction of our model. The central prediction of our model is banks borrowing from DW are fundamentally weaker than banks borrowing from TAF, which in turn are weaker than banks who borrow from neither facilities. This will be the main hypothesis throughout the empirical analysis. Specifically, we conduct two tests. First, we study the correlation between banks' fundamentals and their borrowing decisions. Second, we apply an event-study approach and examine how the market reacted to these borrowing decisions.

4.1 Data and Summary

Throughout the empirical exercise, we combine several datasets. The first one is obtained through Bloomberg and includes 407 institutions that borrowed from the Federal Reserve between Aug 1, 2007 and Apr 30, 2010. These data were released by the Fed on Mar 31, 2011, under a court order, after Bloomberg filed a lawsuit against Fed board for information disclosure.[4] The data contain information on each institution's daily outstanding balance of its borrowing from the discount window, the Term Auction Facility as well as five other related programs.

[4]For details, see https://www.bloomberg.com/news/articles/2011-03-31/federal-reserve-releases-discount-window-loan-records-under-court-order.

Table 1 provides summary statistics. The borrowing institutions are mostly banks ($\approx$ 73%), together with diversified financial services (mostly asset management firms), insurance companies, savings and loans, and other financial service firms. Foreign banks who borrowed through their U.S. subsidiaries were also included. Among them, 92 borrowers were foreign banks who borrowed mainly through their U.S. subsidiaries. Banks' choices of borrowing facilities were quite heterogeneous. While a majority (260 out of 407) tapped both facilities, some only used one throughout the period. The total borrowing events also exhibit sharp heterogeneity: some banks never tapped the discount window, whereas one bank (Alaska USA Federal Credit Union) used it a total of 242 times. Among the 60 TAF auctions, Mitsubishi UFJ Financial Group borrowed a total of 28 times. On average, TAF offered more liquidity (3174 million) than DW (1529 million), consistent with the evidence in Figure 1a. However, the Dexia Group, the bank that borrowed the most from DW took out a total of approximately $190 billion loans over the three-year period, exceeding its counterpart in TAF ($\approx$ $100 billion by Bank of America Corp). This evidence also suggests that DW banks were in more need of liquidity than TAF banks.

Table 1: Summary Statistics of Bloomberg

	N	Mean	Max	Min	SD	10^{th}	50^{th}	90^{th}
No. of Borrowers	407							
Banks	313							
Diversified Financial Services	24							
Insurance Companies	12							
Savings and Loans	30							
Market Cap on Aug 1, 2007 (MM)		28525	399089	11	49876.8	107	7331	81813
Foreign Banks	92							
DW-only banks	18							
TAF-only banks	86							
borrow both	260							
Total DW events		12	242	0	28.7	0	2	35
Total TAF events		5	28	0	5.1	0	3	13
Total DW amount (MM)		1529	190155	0	10393.8	0	20	1809
Total TAF amount (MM)		3174	100167	0	10727.5	0	58	7250
Number of days in debt to Fed		323	814	28	196.8	85	306	606

Our second dataset provides details on all 60 TAFs, including names of bidders (both winners and losers), their bidding rates, the amount awarded, as well as the collaterals pledged to back these loans. We obtain this data by filing a Freedom of Information Act (FOIA) request to the Federal Reserve. Table 2 describes the summary statistics. A total of 434 banks have submitted their bids in TAF. Among them, 22 were classified as Global systemically important (G-SIBs), and 82 were foreign. Indeed, G-SIBs and foreign banks made on average more bids than the rest of the sample, consistent with the existing evidence that that their liquidity positions could be in bigger troubles (Benmelech, 2012). Similar to DW borrowing, banks bidding decisions are also highly skewed: while the median bank submitted a total of eight bids, the most aggressive bank – Mitsubishi UFJ Financial Group – submitted a total of 95 bids through its New York Branch.

Table 2: Summary Statistics of TAF

	N	Mean	Max	Min	SD	10^{th}	50^{th}	90^{th}
No. of Banks	434							
No. of G-SIBs	22							
No. of Foreign Banks	82							
All Banks: no. of bids		13	95	1	13.9	1	8	35
G-SIBs		27	95	1	24.5	1	25	57
Foreign Banks		25	95	1	18.5	4	23	50
All: high-haircut collaterals		0.19	1.00	0.00	0.3	0.00	0.00	0.79
All: low-rating collaterals		0.19	1.00	0.00	0.3	0.00	0.00	0.72

Figure 3 and 4 offer a snapshot of in 60 TAF auctions. All the auctions before Lehman Brothers filed for bankruptcy (23 out of 60) were over-subscribed, meaning that the total bidding amount exceeding the total amount of liquidity provided in the auction. After that, the Federal Reserve increased the offering amount from \$20bn to \$150bn and subsequently, all the auctions were under-subscribed. Banks' bidding amount also vary across all the 60 auctions. In all the 60 auctions, there were banks that submitted the minimum bidding amount (\$5 million) whereas across most auctions until May 4, 2009, some bank submitted the maximum bidding amount (15% of the total auction amount). This observation is consistent with the general impression that the stress in interbank lending market gets much more relieved after mid 2009. The stress in the interbank market is also revealed prominently in their bidding rates. Figure 4 shows the highest bidding rate surged up to 10% when Lehman Brother declared bankruptcy and subsequently gets releaved.

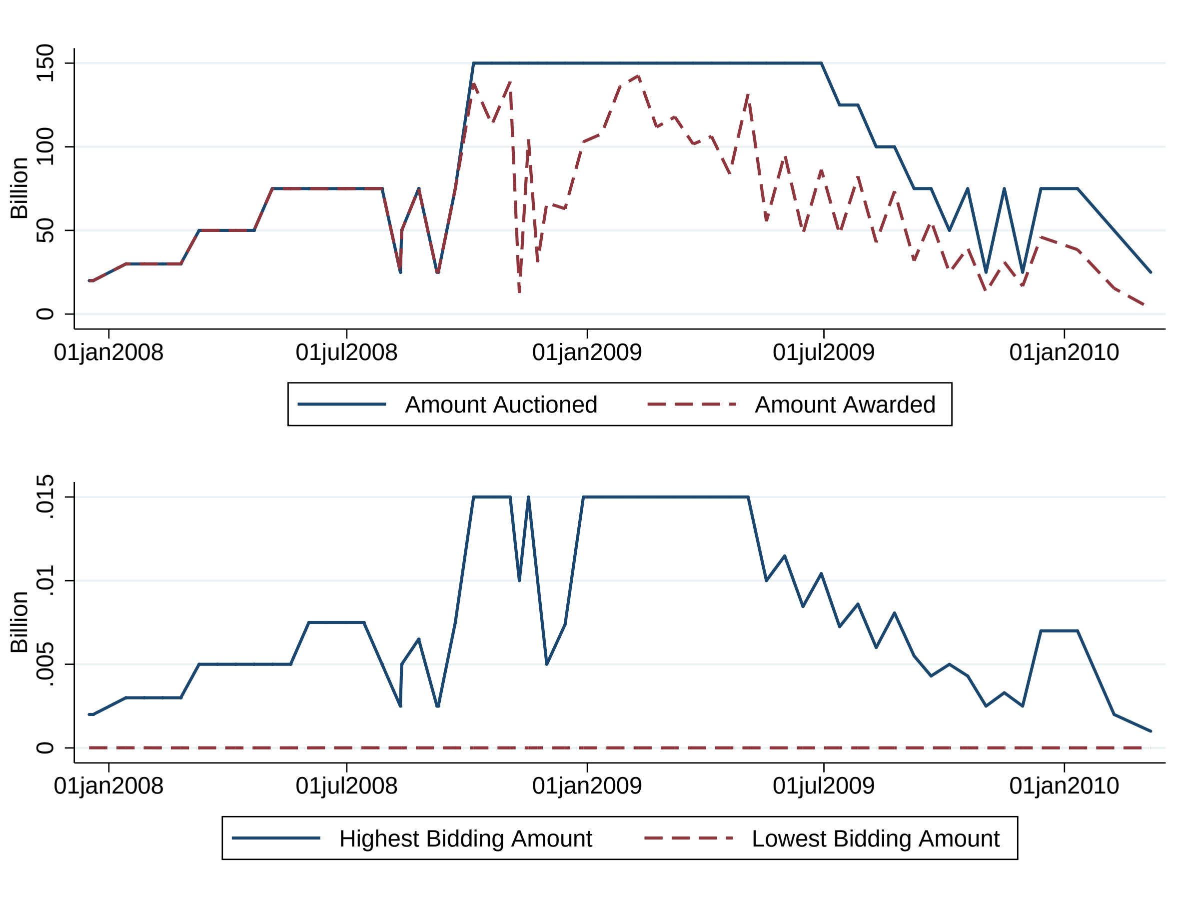


Figure 3: TAF Auction Amount

Finally, we merged both the Bloomberg and the TAF data with each borrower's daily stock market returns as well as various proxies for banks' health (tier-1 capital and liquid assets) obtained from the Bank Regulatory database (Y-9C) at the Band Holding Companies level.

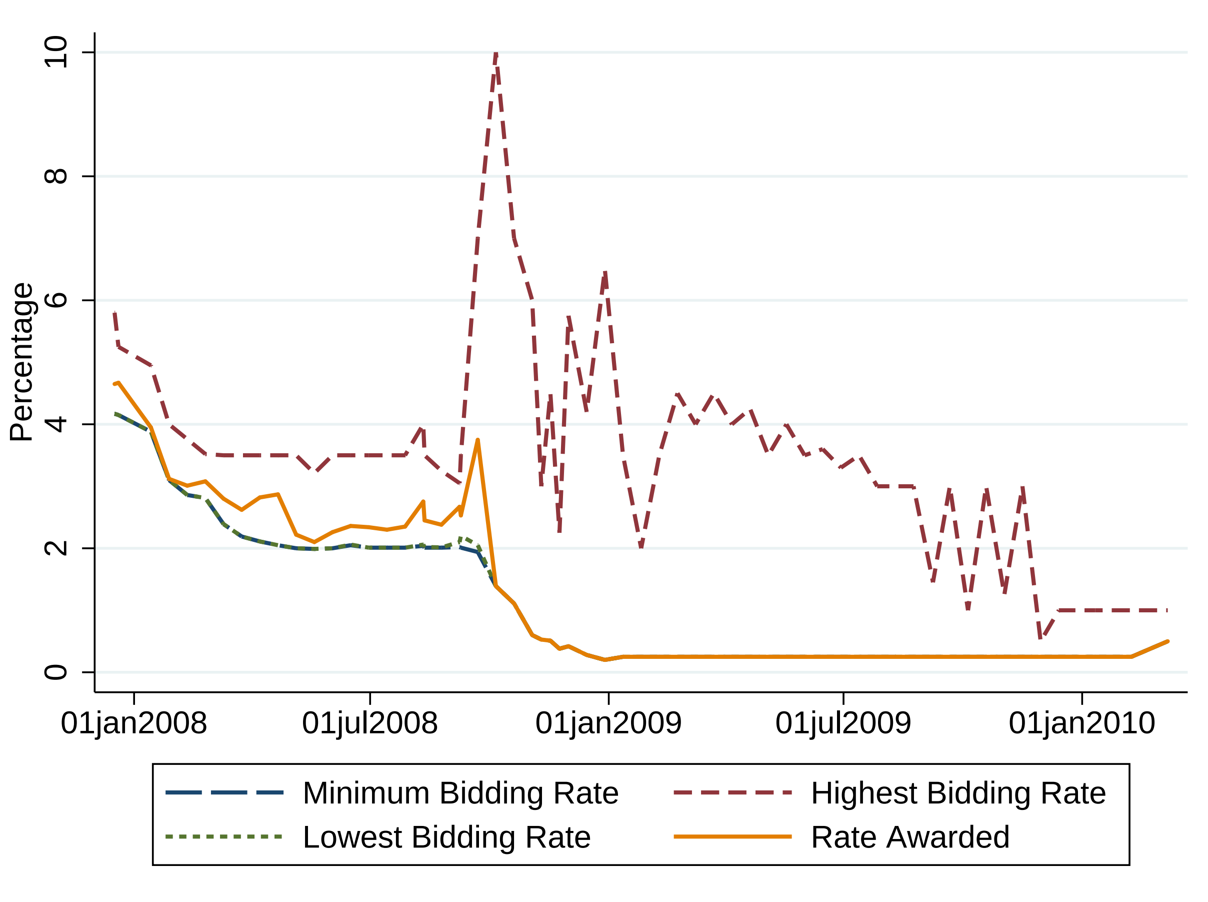


Figure 4: TAF Auction Rate

4.2 A Comparison of DW-banks and TAF-banks

Were banks borrowing from the discount window fundamentally different from banks borrowing from TAF? To answer this question, we conduct the following econometric analysis

$$\frac{\mathrm{DW}_{it}}{\mathrm{DW}_{it} + \mathrm{TAF}_{it}} = a_i + b \times \mathrm{T1CAR}_{it} + c \times S_{it} + Q_t + \varepsilon_i, \tag{3}$$

where the left-hand-side variable, $\frac{\mathrm{DW}_{it}}{\mathrm{DW}_{it}+\mathrm{TAF}_{it}}$, is the share of bank i's total discount window borrowing in quarter t, divided by the sum of total discount window and TAF borrowing within the same period. On the right-hand-side, we control for banks' fixed effects, quarter fixed effects and tier-1 capital ratio, defined as a bank's tier 1 capital divided by its total risk weighted assets. Tier-1 capital ratio is the most commonly-used measurement for bank's financial health. Note that we use the contemporaneous measurement, reflecting the idea that financial strength (θ in the model) is unobservable by market participants. S_{it} is the size of bank i's total asset in quarter t.

Table 3: Borrowing Decision and Tier-1 Capital Ratio

	(1)	(2)	(3)
T1CAR	-3.068***	-2.105*	-2.322***
	(0.770)	(1.170)	(0.790)
log (total asset)	-0.053***	-0.747***	-0.062***
	(0.009)	(0.194)	(0.009)
Foreign	0.271**		0.295**
	(0.133)		(0.130)
Constant	1.652***	12.462***	2.325***
	(0.174)	(3.034)	(0.226)
borrower FE	No	Yes	No
time FE	No	No	Yes
N	561	561	561
R^2	0.077	0.044	0.145

Table 3 report the regression results. Column (1) - (3) differ in the type of fixed-effects controlled in the regression. On average, a 1% increase in an average bank's tier-1 capital ratio reduces the total amount of DW borrowing by 3%. The effect is significant and gets mitigated once bank fixed-effects are taken into account, potentially due to the fact that financial strength is only partially captured by the a bank's tier-1 capital ratio. Across all columns, it is clear that larger banks borrower more from TAF than from DW, consistent with the widely-held perception that they were more concerned with the discount window stigma. Moreover, foreign-banks borrow dis-proportionally more from DW than from TAF.

4.3 Market Response

How did market respond to borrowings from different programs? In this subsection, we conduct an event-study analysis following each borrowing event and study how the stock price changes. Specifically, the estimation window is set as the period before the interbank market froze up: Jan 3, 2005 to Aug 1, 2007. Predicted returns are estimated using both the market model. We choose the length of the event window as give days after the borrowing event, based on the assumption that one source of detection is the weekly public report of aggregate DW borrowings. Table 4 reports the five-day cumulative abnormal returns (CAR)

Table 4: CAR following Borrowing Events

	(1) DW	(2) DW/TAF	(3) TAF/DW	(4) TAF
CAR	-0.009***	-0.015**	0.004	-0.005
	(0.002)	(0.008)	(0.007)	(0.004)
N	2948	209	257	720

following each borrowing event. Column (1) shows for an average bank, its stock price declines by abot 0.9% in the subsequent five days following a DW borrowing. In contrast, Column (4) shows the CAR is only 0.5% (also not significant) following a TAF borrowing. In Column (2) and (3), we classify the DW borrowing event into two groups. The first group borrowed from the discount window within the three-day window before a TAF auction. Presumably, this group of banks were desperate for liquidity and if detected, the cost of stigma should be the highest. The second group is comprised of banks who tapped discount window shortly after the auction. According to our model, their liquidity conditions should be further stronger. Column (2) shows that the CAR is even more negative for banks who borrowed DW shortly before TAF, whereas according to Column (3), the CAR is no longer significant for banks who borrowed DW shortly after TAF. These results are consistent with our model's prediction that on average, DW borrowing carries a higher stigma cost, and such cost is even higher if a bank tapped DW shortly before a TAF were to be held. One caveat though, is that the real world is dynamic, whereas our model only spans two periods. Consequently, banks who borrowed from the discount window shortly after TAF might also be those who were newly hit with liquidity shocks. If we take this effect into account, the comparison between the second and the last group will be ambiguous.

5 Conclusion

In this paper, we investigated how the Term Auction Facility mitigates the stigma associated with borrowing from the Discount Window. We constructed an auction model with endogenous participation and showed optimal auction bidding strategies that internalized any stigma associated with the auction naturally increased participate and consequently mitigated the borrowing stigma.

We showed the following results consistent with the empirical observations. First, banks with strong financial health were reluctant to borrow from the Discount Window due to the standard adverse selection logic a-la Akerlof (1970). Second, when both DW and TAF are available, the weakest banks borrowed from the DW, and relatively strong ones participated in TAF. Among those who lost in the auction, relatively weak ones moved on to borrow from the DW. Third, we show the introduction of TAF may or may not expand the set of banks who obtained liquidity. Lastly, our model suggests the stop-out rate of TAF may be higher or lower than the primary discount rate.

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A Appendix

A.1 Detailed Expressions of the Equilibrium

A.1.1 Discount Window only

Welfare In this case, an individual bank's expected utility is

$$U = \int_0^{\theta_1} \left[(1-\theta) R - r_D - k_D \right] f(\theta)\, d\theta + \int_{\theta_1}^{1} (-k_N) f(\theta)\, d\theta.$$

Under uniform distribution,

$$U = \left(1 - \frac{\theta_1}{2}\right) R\theta_1 - (r_D + k_D)\,\theta_1 - k_N (1-\theta_1).$$

The total welfare of all banks are

$$W = nU.$$

A.1.2 TAF only

The equilibrium is characterized by two thresholds $\{\theta_2, \theta_A\}$:

$$(1-\theta_2) R - r_D - k_D - \delta = -k_N \tag{4}$$

$$(1-\theta_A) R - (k_A - k_N) - \delta = r_A. \tag{5}$$

1. Banks between $[0, \theta_A]$ bid in TAF. Those in $[0, \theta_2]$ make identical bids $\beta^D(\theta) = r_D + (k_D - k_A)$ and those in $[\theta_2, \theta_A]$ make bids $\beta^N(\theta) = (1-\theta) R - (k_A - k_N) - \delta$.

2. Among banks who lose in TAF, those between $[0, \theta_2]$ borrow from discount window.

Let us write the expressions for equilibrium penalties: k_D, k_A and k_N. Clearly,

$$k_D = \int_0^{\theta_2} \frac{k(\theta)}{F(\theta_2)} dF(\theta).$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_D = p\left(1 - \frac{\theta_2}{2}\right).$$

Next, we compute k_A. In this case, let τ be the $p.d.f$ for the m^{th} lowest type among all the n banks and $h(\tau)$ be its $p.d.f$.

$$h(\tau) = \binom{n}{m}\binom{m}{1}[F(\tau)]^{m-1}[1-F(\tau)]^{n-m} f(\tau).$$

If $\tau \in [0, \theta_2]$, then banks who borrow from TAF follow the same distribution as $F(\theta)$ on $[0, \theta_2]$, which leads to the first component of k_A:

$$k_{A1} = \int_0^{\theta_2} \mathbb{E}\left[k(\theta)\,\middle|\theta \in [0, \theta_2]\right] h(\tau)\, d\tau.$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_{A1} = \int_0^{\theta_2} p\left(1 - \frac{1}{2}\theta_2\right) h(\tau)\, d\tau.$$

If $\tau \in [\theta_2, \theta_A]$, then exactly m banks receive liquidity from TAF, and one of them is τ, with the other $m-1$ banks follow the same distribution of $F(\theta)$ on $[0, \tau]$. In this case,

$$k_{A2} = \int_{\theta_2}^{\theta_A} \left\{\frac{1}{m}k(\tau) + \frac{m-1}{m}\mathbb{E}\left[k(\theta)\,\middle|\theta \in [0, \tau]\right]\right\} h(\tau)\, d\tau$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_{A2} = \int_{\theta_2}^{\theta_A} \left\{\frac{1}{m}p(1-\tau) + \frac{m-1}{m}p\left(1 - \frac{\tau}{2}\right)\right\} h(\tau)\, d\tau.$$

If $\tau \in [\theta_A, 1]$, then all banks who bid in TAF will receive liquidity. In this case,

$$k_{A3} = \int_{\theta_A}^{1} \mathbb{E}\left[k(\theta)\,\middle|\theta \in [0, \theta_A]\right] h(\tau)\, d\tau.$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_{A3} = \int_{\theta_A}^{1} p\left(1 - \frac{1}{2}\theta_A\right) h(\tau)\, d\tau.$$

The stigma cost associated with TAF borrowing is therefore

$$k_A = k_{A1} + k_{A2} + k_{A3}.$$

Now we compute k_N. Let us first consider the case that the m^{th} lowest bank is below θ_2. In this case, banks who do not necessarily borrow are those who falls into $[\theta_2, 1]$, which leads to the first component of k_N:

$$k_{N1} = \int_{0}^{\theta_2} \mathbb{E}\left[k\left(\theta \middle| \theta \in [\theta_2, 1]\right)\right] h(\tau)\, d\tau.$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_{N1} = \int_{0}^{\theta_2} p\left(1 - \frac{\theta_2 + 1}{2}\right) h(\tau)\, d\tau.$$

Next, consider the case when τ falls into $[\theta_2, \theta_A]$:

$$k_{N2} = \int_{\theta_2}^{\theta_A} \mathbb{E}\left[k(\theta) \left| \theta \in [x, 1]\right.\right] \binom{n}{m} \binom{m}{1} F^{m-1}(x) \left[1 - F(x)\right]^{n-m} f(x)\, dx.$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_{N2} = \int_{\theta_2}^{\theta_A} p\left(1 - \frac{x+1}{2}\right) \binom{n}{m} \binom{m}{1} x^{m-1} \left[1 - x\right]^{n-m} dx.$$

Finally, when τ falls into $[\theta_A, 1]$:

$$k_{N3} = \left[1 - H(\theta_A)\right] \mathbb{E}\left[k(\theta) \left| \theta \in [\theta_A, 1]\right.\right].$$

[5] With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_{N3} = p\left(1 - \frac{\theta_A + 1}{2}\right)\left[1 - H(\theta_A)\right].$$

The stigma cost associated with non-borrowing is therefore

$$k_N = k_{N1} + k_{N2} + k_{N3}.$$

Welfare Let $u(\theta)$ be the an individual expected utility given its type θ. Let

$$g(\tau) = (n-1)\binom{n-2}{m-1}\left[F(\tau)\right]^{m-1}\left[1 - F(\tau)\right]^{n-m-1} f(\tau).$$

τ is the m^{th} lowest bank among all the other $n-1$ banks.

1. If $\theta \in [0, \theta_2]$, then

$$\begin{aligned} u(\theta) &= \int_0^{\theta_2}\left[(1-\theta)R - r_D - k_D - \delta\right] g(\tau)\, d\tau + \int_{\theta_2}^{\theta_A}\left[(1-\theta)R - (1-\tau)R - k_N\right] g(\tau)\, d\tau \\ &\quad + \left[1 - G(\theta_A)\right]\left[(1-\theta)R - r_A - k_A - \delta\right]. \end{aligned}$$

2. If $\theta \in [\theta_2, \theta_A]$, then

$$\begin{aligned} u(\theta) &= \int_0^{\theta}(-k_N)\, g(\tau)\, d\tau + \int_{\theta}^{\theta_A}\left[(1-\theta)R - (1-\tau)R - k_N\right] g(\tau)\, d\tau \\ &\quad + \left[1 - G(\theta_A)\right]\left[(1-\theta)R - r_A - k_A - \delta\right]. \end{aligned}$$

3. If $\theta \in [\theta_A, 1]$, then

$$u(\theta) = -k_N.$$

An individual bank's expected welfare is

$$U = \int_0^1 u(\theta) f(\theta)\, d\theta.$$

[5] I used to write $k_{N3} = \int_{\theta_A}^1 \mathbb{E}\left[k(\theta)\,\middle|\,\theta \in [\theta_A, 1]\right]\binom{n}{m}\binom{m}{1}F^{m-1}(x)\left[1 - F(x)\right]^{n-m} f(x)\, dx$ but this is not correct. It fails to take into account that exactly one bank's type is τ. The results are identical though.

Under uniform distribution, this simplifies to

A.1.3 Both Discount Window and TAF

The equilibrium is characterized by two thresholds $\{\theta_D, \theta_A\}$:

1. Banks between $[0, \theta_D]$ borrow from DW1.
2. Banks between $[\theta_D, \theta_A]$ bid in the auction. They bid exactly $\beta^N(\theta) = (1-\theta)R - (k_A - k_N) - \delta$.

$$\begin{aligned} k_A + \delta + \int_{\theta_D}^{\theta_A} \beta(\tau) g(\tau) d\tau + (1 - G(\theta_A)) r_A &= r_D + k_D \quad (6) \\ (1-\theta_A) R - (k_A - k_N) - \delta &= r_A \quad (7) \end{aligned}$$

where $g(\tau) = (n-1)\binom{n-2}{m-1}(F(x) - F(\theta_D))^{m-1}(1 - F(x) + F(\theta_D))^{n-m-1} f(\tau)$ is the $p.d.f.$ of the highest losing type from the perspective of a type θ_D bank.

Let us write the expressions for equilibrium penalties: k_D, k_A and k_N. Clearly,

$$k_D = \int_0^{\theta_D} \frac{k(\theta)}{F(\theta_D)} dF(\theta).$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_D = p\left(1 - \frac{\theta_D}{2}\right).$$

Next, we compute k_A. In this case, let τ be the the m^{th} lowest type among all the banks that fall into $[\theta_D, \theta_A]$ and $h(\tau)$ be its $p.d.f.$

$$h(\tau) = \binom{n}{m}\binom{m}{1}[F(\tau) - F(\theta_D)]^{m-1}[1 - F(\tau) + F(\theta_D)]^{n-m} f(\tau).$$

With probability $h(\tau)$, the penalty shall be $\mathbb{E}\left[k(\theta) \left| \theta \in [\theta_D, \tau]\right.\right]$. Otherwise, the penalty shall be $\mathbb{E}\left[k(\theta) \left| \theta \in [\theta_D, \theta_A]\right.\right]$. Therefore

$$k_A = \int_{\theta_D}^{\theta_A} \mathbb{E}\left[\frac{1}{m}k(\tau) + \frac{m-1}{m}k(\theta) \left| \theta \in [\theta_D, \tau]\right.\right] h(\tau) d\tau + \left[1 - \int_{\theta_D}^{\theta_A} h(\tau) d\tau\right] \mathbb{E}\left[k(\theta) \left| \theta \in [\theta_D, \theta_A]\right.\right].$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_A = \int_{\theta_D}^{\theta_A} \left[\frac{1}{m} p(1-\tau) + \frac{m-1}{m} p \left(1 - \frac{1}{2}(\theta_D + \tau) \right) \right] h(\tau)\, d\tau + \left[1 - \int_{\theta_D}^{\theta_A} h(\tau)\, d\tau \right] p \left(1 - \frac{1}{2}(\theta_D + \theta_A) \right).$$

Next, we consider k_N. For $\tau \in [\theta_D, \theta_A]$, with probability $h(\tau)$, the penalty shall be $\mathbb{E}\left[k(\theta) \,|\theta \in [\tau, 1]\right]$. Otherwise, the penalty shall be $\mathbb{E}\left[k(\theta) \,|\theta \in [\theta_A, 1]\right]$. Therefore

$$k_N = \int_{\theta_D}^{\theta_A} \mathbb{E}\left[k(\theta) \,|\theta \in [\tau, 1]\right] h(\tau)\, d\tau + \left[1 - \int_{\theta_D}^{\theta_A} h(\tau)\, d\tau \right] \mathbb{E}\left[k(\theta) \,|\theta \in [\theta_A, 1]\right].$$

With uniform distribution and $k(\theta) = p(1-\theta)$,

$$k_N = \int_{\theta_D}^{\theta_A} p \left(1 - \frac{1}{2}(1+\tau) \right) h(\tau)\, d\tau + \left[1 - \int_{\theta_D}^{\theta_A} h(\tau)\, d\tau \right] p \left(1 - \frac{1}{2}(\theta_A + 1) \right).$$

Welfare Let

$$g(\tau) = (n-1) \binom{n-2}{m-1} \left[F(\tau) - F(\theta_D)\right]^{m-1} \left[1 - F(\tau) + F(\theta_D)\right]^{n-m-1} f(\tau).$$

1. If $\theta \in [0, \theta_D]$, then

$$u(\theta) = (1-\theta) R - r_D - k_D.$$

2. If $\theta \in [\theta_D, \theta_A]$, then

$$\begin{aligned} u(\theta) &= \int_{\theta_D}^{\theta} (-k_N)\, g(\tau)\, d\tau + \int_{\theta}^{\theta_A} \left[(1-\theta) R - \beta^N(\tau) - k_A - \delta\right] g(\tau)\, d\tau \\ &\quad + \left[1 - G(\theta_A)\right] \left[(1-\theta) R - r_A - k_A - \delta\right]. \end{aligned}$$

3. If $\theta \in [\theta_A, 1]$, then

$$u(\theta) = -k_N.$$

An individual bank's expected welfare is

$$U = \int_0^1 u(\theta) f(\theta)\, d\theta.$$

Next, we compute the aggregate welfare. Let M be the number of banks whose realized types are in $[\theta_D, \theta_A]$. The aggregate welfare if $M \leq m$ differs from that if $M > m$. Clearly, the probability that a total of M banks fall into $[\theta_D, \theta_A]$ is

$$P\left(M, F\left(\theta_A\right) - F\left(\theta_D\right), n\right) = \binom{n}{M} \left[F\left(\theta_A\right) - F\left(\theta_D\right)\right]^M \left[1 - F\left(\theta_A\right) + F\left(\theta_D\right)\right]^{n-M}.$$

- Subcase 1: $M \leq m$. The aggregate welfare is

$$W_M = (n - M) \frac{\left[\int_0^{\theta_D} u(\theta) f(\theta) \, d\theta + \int_{\theta_A}^1 u(\theta) f(\theta) \, d\theta\right]}{1 - F\left(\theta_A\right) + F\left(\theta_D\right)} + M \frac{\int_{\theta_D}^{\theta_A} \left[(1-\theta) R - r_A - k_A - \delta\right] f(\theta) \, d\theta}{F\left(\theta_A\right) - F\left(\theta_D\right)}.$$

- Subcase 2: $M > m$. Let τ be the $m + 1^{th}$ lowest bank among the M banks. Its distribution is

$$\begin{aligned} h(\tau) &= \binom{M-1}{m} \left[\frac{F(\tau) - F\left(\theta_D\right)}{F\left(\theta_A\right) - F\left(\theta_D\right)}\right]^m \left[1 - \frac{F(\tau) - F\left(\theta_D\right)}{F\left(\theta_A\right) - F\left(\theta_D\right)}\right]^{M-1-m} \frac{f(\tau)}{F\left(\theta_A\right) - F\left(\theta_D\right)} \\ &= P\left(m, \frac{F(\tau) - F\left(\theta_D\right)}{F\left(\theta_A\right) - F\left(\theta_D\right)}, M - 1\right) \frac{f(\tau)}{F\left(\theta_A\right) - F\left(\theta_D\right)}. \end{aligned}$$

  The aggregate welfare is

$$\begin{aligned} W_M &= (n - M) \frac{\left[\int_0^{\theta_D} u(\theta) f(\theta) \, d\theta + \int_{\theta_A}^1 u(\theta) f(\theta) \, d\theta\right]}{1 - F\left(\theta_A\right) + F\left(\theta_D\right)} + (M - m)\left(-k_N\right) \\ &\quad + m \frac{\int_{\theta_D}^{\theta_A} \int_{\theta_D}^{\theta_A} \left[(1-\theta) R - \beta^N(\tau) - k_A - \delta\right] g(\tau) \, d\tau f(\theta) \, d\theta}{F\left(\theta_A\right) - F\left(\theta_D\right)} \end{aligned}$$

The total welfare is

$$W = \sum_{M=0}^{n} P\left(M, F\left(\theta_A\right) - F\left(\theta_D\right), n\right) W_M.$$

A.2 Proofs

A.2.1 Proofs in Subsection 3.1

Proof of Proposition 1

Proof. A type-θ bank borrows from discount window against not borrowing at all if and only if

$$u_D = (1-\theta) R - r_D - k_D \geq 0.$$

Clearly, the incentive to borrow from discount window decreases with θ. Therefore, for any given k_D, a bank borrows from discount window if and only if it type satisfies

$$\theta \leq \frac{R - r_D - k_D}{R}.$$

Bank θ^{DW} is indifferent between borrowing, implying that

$$\left(1-\theta^{DW}\right) R - r_D = k_D.$$

Specifically, θ^{DW} is determined by

$$\left(1-\theta^{DW}\right) R - r_D = \frac{\int_0^{\theta^{DW}} k(\theta) f(\theta) d\theta}{F(\theta^{DW})}.$$

$\square$

Proof of Proposition 2

Proof. In this case, all banks bid their WLP, which equal to

$$\beta(\theta) = (1-\theta) R - k_A - \delta.$$

Bank θ_{TAF} bids exactly up to reserve rate r_A:

$$\theta_{TAF} = 1 - \frac{k_A + \delta + r_A}{R}.$$

The unique solution is

$$\left(1-\theta^{TAF}\right) R - r_A - \delta = \int_0^{\theta^{TAF}} \frac{k(\theta) f(\theta) d\theta}{F(\theta^{TAF})}.$$

$\square$

A.2.2 Proofs in Subsection 3.2

Proof of Lemma 2

Proof. In the auction, the winning bank pays the highest loser's bid. Therefore, its own bid does not affect its equilibrium payments, only its chances of winning the auction. Therefore, it is its dominant strategy to bid its own willingness to pay.

Bank θ's willingness to pay satisfies

$$(1-\theta) R - \beta(\theta) - k_A - \delta = \max\{(1-\theta) R - r_D - k_D - \delta, -k_N\}.$$

If $(1-\theta) R - r_D - k_D - \delta \geq -k_N$ so that the losing bank will go to the discount window, then

$$\beta(\theta) = \beta^D(\theta) = r_D + (k_D - k_A).$$

Otherwise,

$$\beta(\theta) = \beta^N(\theta) = (1-\theta) R - (k_A - k_N) - \delta.$$

$\square$

Proof of Lemma 1

Proof. A bank borrows from discount window during period 2 if and only if

$$\begin{aligned} u_2 = (1-\theta) R - r_D - k_D - \delta &\geq u_N = -k_N \\ \theta &\leq \theta_2(k_D, k_N) \equiv 1 - \frac{r_D + k_D - k_N + \delta}{R}. \end{aligned}$$

$\square$

Proof of Proposition 3

Proof. Clearly,

$$u_1(\theta) = (1-\theta) R - r_D - k_D.$$

Let $\tau \in [0,1]$ be the highest losing bank and $H(\tau)$ be its distribution. Let us first consider $u_A(\theta)$ for $\theta < \theta_2$. If $\tau < \theta_2$, bank θ's payoff from winning the auction is $(1-\theta) R - \beta^D(\theta) - k_A - \delta$, which simplifies to $(1-\theta) R - r_D - k_D - \delta$. If it loses, it turns to discount window again and receives the same payoff $(1-\theta) R - r_D - k_D - \delta$ as well. If $\tau \geq \theta_2$, a bank $\theta < \theta_2$ wins the auction for sure and receives payoff $(1-\theta) R - \beta^N(\tau) - k_A - \delta$, which simplifies to $(1-\theta) R - (1-\tau) R - k_N$. Therefore,

$$u_A(\theta) = [(1-\theta) R - r_D - k_D - \delta] H(\theta_2) + \int_{\theta_2}^{1} [(1-\theta) R - (1-\tau) R - k_N] \, dH(\tau) \quad \text{if } \theta < \theta_2.$$

Next, we consider $u_A(\theta)$ for $\theta > \theta_2$. In this case, a bank θ receives $(1-\theta) R - (1-\tau) R - k_N$ if it wins in the auction $(\tau > \theta)$. If it loses, it receives $-k_N$. Therefore,

$$u_A(\theta) = \int_0^{\theta} (-k_N) \, dH(\tau) + \int_{\theta}^{1} [(1-\theta) R - (1-\tau) R - k_N] \, dH(\tau) \quad \text{if } \theta \geq \theta_2.$$

Taking the difference, we have

$$u_1(\theta) - u_A(\theta) = \begin{cases} \delta + \int_{\theta_2}^{1} [(\theta_2 - \tau) R] \, dH(\tau) & \text{if } \theta < \theta_2 \\ [(\theta_2 - \theta) R + \delta] + \int_{\theta}^{1} (\theta - \tau) R dH(\tau) & \text{if } \theta \geq \theta_2. \end{cases}$$

Clearly, $u_1(\theta) - u_A(\theta)$ is continuous and stays at a positive constant when $\theta < \theta_2$. When $\theta > \theta_2$, it is easily checked that

$$\frac{d\left(u_1(\theta) - u_A(\theta)\right)}{d\theta} = -H(\theta) R < 0.$$

□

Proof of Theorem 1

Proof. Let $h_m^n(x) \equiv \binom{n}{m} x^m (1-x)^{n-m}$. Define three correspondences:

$$\phi_1(\theta_1, \theta_2, \theta_A) = \left\{ \theta : u_1(\theta|\theta_1, \theta_2, \theta_A) - \max\{u_A(\theta|\theta_1, \theta_2, \theta_A), u_N(\theta|\theta_1, \theta_2, \theta_A)\} \geq 0 \right\} \cup \{0\},$$

$$\phi_2(\theta_1,\theta_2,\theta_A) = \left\{\theta : u_2(\theta|\theta_1,\theta_2,\theta_A) - u_N(\theta|\theta_1,\theta_2,\theta_A) \geq 0\right\} \cup \{0\},$$

and

$$\phi_A(\theta_1,\theta_2,\theta_A) = \left\{\theta : u_A(\theta|\theta_1,\theta_2,\theta_A) - u_N(\theta|\theta_1,\theta_2,\theta_A) \geq 0\right\} \cup \{0\},$$

where

$$u_1(\theta|\theta_1,\theta_2,\theta_A) = (1-\theta)R - r_D - k_D(\theta_1,\theta_2,\theta_A),$$

$$u_2(\theta|\theta_1,\theta_2,\theta_A) = (1-\theta)R - r_D - k_D(\theta_1,\theta_2,\theta_A) - \delta,$$

$$u_A(\theta|\theta_1,\theta_2,\theta_A) = \text{(note: wrong)}$$

$$\begin{cases} (1-\theta)R - \int_{\theta_1}^{1} \left[\max(\beta(\tau), r_A) - k_A(\theta_1,\theta_2,\theta_A)\right] dh_m^{n-1}\left[F(\tau)-F(\theta_1)\right] - \delta & 0 \leq \theta \leq \theta_1 \\ \int_{\theta}^{1} \left[(1-\theta)R - \max(\beta(\tau), r_A) - k_A(\theta_1,\theta_2,\theta_A)\right] dh_m^{n-1}\left[F(\tau)-F(\theta_1)\right] - \delta & \\ + \int_{\theta_1}^{\theta} \left[-k_N(\theta_1,\theta_2,\theta_A)\right] dh_m^{n-1}\left[F(\tau)-F(\theta_1)\right] & \theta_1 \leq \theta \leq \theta_A \end{cases},$$

and

$$u_N(\theta|\theta_1,\theta_2,\theta_A) = -k_N(\theta_1,\theta_2,\theta_A).$$

Economically, if it is believed that (i) $[0,\theta_1]$ is the set of banks willing to borrow from discount window 1, (ii) $[0,\theta_A]$ is the set of banks willing to bid if it has not borrowed from discount window 1, and (iii) $[0,\theta_2]$ is the set of banks willing to borrow from discount window 2 if it has not borrowed after auction, then optimally, (i) $\phi_1(\theta_1,\theta_2,\theta_A)$ is the set of banks willing to borrow from discount window 1, (ii) $\phi_A(\theta_1,\theta_2,\theta_A)$ is the set of banks willing to bid in the auction if it has not borrowed from discount window 1, and (iii) $\phi_A(\theta_1,\theta_2,\theta_A)$ is the set of banks willing to borrow from discount window 2 if it has not borrowed after auction. We have an equilibrium if the belief is consistent with the optimal action: $[0,\theta_1] = \phi_1(\theta_1,\theta_2,\theta_A)$, $[0,\theta_2] = \phi_2(\theta_1,\theta_2,\theta_A)$, and $[0,\theta_A] = \phi_A(\theta_1,\theta_2,\theta_A)$; or more simply, if $(\theta_1,\theta_2,\theta_A) \in \phi(\theta_1,\theta_2,\theta_A) \equiv (\phi_1(\theta_1,\theta_2,\theta_A), \phi_2(\theta_1,\theta_2,\theta_A), \phi_A(\theta_1,\theta_2,\theta_A))$. Hence, to prove the existence of an equilibrium, it suffices to show that the correspondence $\phi \equiv (\phi_1,\phi_2,\phi_A)$ has a fixed point.

Mathematically, each of the three correspondences is well-defined on $X \equiv [0,1]^3 \cap$

$\{(\theta_1, \theta_2, \theta_A) : \theta_1 \leq \theta_A\}$, a non-empty, compact, and convex subset of the Euclidean space $\mathbb{R}^3$, and is upperhemicontinuous with the property that $\phi_\omega(x)$ for each $\omega \in \{1, 2, A\}$ is non-empty, closed, and convex for all $x \in X$. By Kakutani's fixed point theorem, $\phi : X \to 2^X$ has a fixed point $x \in X$. $\square$